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Repelling Conditions for Boundary Sets Using Liapunov-like Functions. II. Persistence and Periodic Solutions

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This paper deals with repulsivity conditions for the points of the boundary of a set G, with respect to the solutions of the differential system $x' = f(t, x)$ which are contained in the set G. Applications are given to the problem of "persistence" (i.e., solutions starting at the interior of a set remain asymptotically far from its boundary) and to the existence of periodic solutions (using the "nonejective" fixed point theorem).

1. INTRODUCTION

Let $f: J \times \Omega \to \mathbb{R}^d$ be a *continuous* function, where $J = [a, b[$ is a nondegenerate real interval ($b \in \mathbb{R} \cup \{+\infty\}$) and $\Omega \subset \mathbb{R}^d$ is a nonempty *open* subset of the d-dimensional real euclidean space $\mathbb{R}^d$ endowed with the usual norm $|\cdot| = (\cdot|\cdot)^{1/2}$.

Throughout the paper, any solution $x(\cdot)$ of the Cauchy Problem

$$x' = f(t, x) \qquad (' = d/dt) \tag{1.1}$$

$$x(t_0) = x_0 \tag{1.2}$$

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is supposed to be noncontinuable (see [20]) and its right maximal interval of existence is denoted by $I_x := [t_0, t_x[$. We notice that, unless when explicitly stated, forward uniqueness for the solutions to (1.1)–(1.2) is not assumed.

Let $N \subset \Omega$ be *closed* relatively to Ω and suppose that N is a *flow-invariant* set with respect to Eq. (1.1); that is, if $x(\cdot)$ is any (noncontinuable) solution to (1.1)–(1.2) with $(t_0, x_0) \in J \times N$, then $x(t) \in N$ for all $t \in I_x$ (of course, the choice $N = \Omega$ is allowed). Finally, let M be a subset of N with $\mathrm{int}_N M$ (the interior of M with respect to N) nonempty.

The aim of our work is to produce conditions ensuring that if $x(\cdot)$ is any solution of (1.1)–(1.2), with $x_0 \in \mathrm{int}_N M$, then $x(t) \in \mathrm{int}_N M$ for all $t \in I_x$ and $x(t)$ remains far away from the relative boundary $\mathrm{fr}_N M$ as t approaches t_x. The motivation of such investigation is related to the so-called problem of persistence arising from population dynamics. Namely, in many concrete applications of differential equations models to ecological or biochemical phenomena, M is a closed subset of the positive orthant $\mathbb{R}^d_+$ and it is important to know that the vector $x(t)$, during its evolution, does not approach extinction states which are usually represented by the points of $\mathrm{fr}_N M$. To this purpose, we'll study the repulsivity of the boundary of M with respect to the solutions of (1.1) which come from $\mathrm{int}_N M$. More generally, for a pair of disjoint subsets G and S of N, with G open in N, we find geometric criteria on the behaviour of $f(\cdot,\cdot)$ in a neighbourhood of S, according to which S repels the trajectories of Eq. (1.1) lying in G. It is clear that in this way we may treat the above mentioned problem of persistence provided that the obvious choice $G = \mathrm{int}_N M$, $S = \mathrm{fr}_N M$ is made, but this general approach allows us also to study the repulsivity of only some pieces of $\mathrm{fr}_N M$ and, therefore, to obtain a wider class of applications, like, for instance, to the periodic boundary value problem.

The proof of the results makes use of families of Liapunov-like functions and differential inequalities. Essentially, some classical facts from stability theory [31] are combined with a generalization of the concept of attractive bound set [16], usually employed for dealing with boundary value problems. Similar ideas have been developed recently in [9, 12]. In [12], with the same technique, we examined various repelling conditions for S in order to obtain, repectively, flow-invariance of $\mathrm{int}_N M$, non-existence results for terminal value problems associated to Eq. (1.1), and weak persistence. From this point of view, the present work completes and concludes the program carried out in [9, 12]. More precisely, compared with [12], we produce here stronger conditions for the repulsivity of S and applications are given to persistence, uniform persistence, and existence of periodic solutions.

The plan of the paper is the following. In Section 2 we obtain repelling conditions for a set S with respect to the solutions of Eq. (1.1) which remain in a set G, where $G \cap S = \emptyset$. Such conditions are then applied in Section 3 for achieving persistence-type theorems relative to a set $M \subset N$. Compared with the results previously appearing in the literature, ours improve some theorems in [18, 9] and provide an analogue for the nonautonomous systems of the results by J. Hofbauer [22] and V. Hutson [25] dealing with autonomous systems. In recent years, many important achievements have been obtained for persistence of autonomous systems [40, 22, 25] or, more generally, semiflows in locally compact metric spaces [5, 6, 13]. On the other hand, we observe that in [40, 22, 25] only one Liapunov function is considered and we also recall that, in [9], we found examples showing that many theorems which hold for autonomous systems are no longer true for Eq. (1.1). Finally, in Section 4, we present an application in which the results of repulsivity of the set S are employed to get the existence of positive periodic solutions to system (1.1). In this example, a nontrivial periodic solution of Eq. (1.1) is obtained by applying the Browder theorem, on the existence of nonejective fixed points [4], to the Poincaré map associated to (1.1). In this way, results by R. E. Gaines and J. Santanilla [17] and J. Santanilla [39] are also sharpened and improved. In particular we show that various technical assumptions required in [17, 39] are actually unnecessary and can be dropped. For a more complete discussion about existence of periodic solutions of (1.1), lying in a conical shell, see also [11] and the references quoted therein.

The paper is as self-contained as possible, even if notation and terminology are borrowed from [12]. For a continuous function $V: J' \times \Omega' \to \mathbb{R}$, with $J' = [a', b'[\subset J$ and $\Omega' \subset \Omega$ (Ω' open), following [46], we set

$$\dot{V}(t, x) := \liminf_{h \to 0^+} [V(t+h, x+hf(t, x)) - V(t, x)]/h.$$

Recall that if V is locally lipschitzian in x and $x(\cdot)$ is a solution of (1.1), then we have (see [46])

$$\dot{V}(t, x(t)) = D_+ V(t, x(t)),$$

where D_+ denotes the lower right Dini derivative.

We denote by $\mathbb{R}^-(\mathbb{R}^+)$ the set of the negative (positive) reals and define $\mathbb{R}_- = \mathbb{R}^- \cup \{0\}$ ($\mathbb{R}_+ = \mathbb{R}^+ \cup \{0\}$). Finally, $\mathrm{dist}(\cdot, \cdot)$ is the (euclidean) matric) in $\mathbb{R}^d$ and $B(x, r)$ and $B[x, r]$ are, respectively, the open and the closed ball of center x and radius r. Analogously, for a set $K \subset \mathbb{R}^d$, we define the r-neighbourhoods (open and closed) of K by

$$B(K, r) := \{ y \in \mathbb{R}^d: \exists x \in K; |y - x| < r \},$$

$$B[K, r] := \{ y \in \mathbb{R}^d: \exists x \in K; |y - x| \leqslant r \}.$$

$\mathrm{cl}_B A$, $\mathrm{int}_B A$, and $\mathrm{fr}_B A$ are, respectively, the closure, interior, and boundary of a set $A \subset B$, relatively to B. The subscript B is omitted when $B = \mathbb{R}^d$.

2. Repelling Sets, Persistence, and Uniform Persistence

In this section we consider conditions of repulsivity of boundary points stronger than those examined in [12]. Such conditions will be used to get *persistence* and *uniform persistence* for system (1.1).

Following [5, 6, 15], we recall that, given a set $M \subset N$, with $\mathrm{int}_N M \neq \emptyset$, we say that system (1.1) is *persistent in N, with respect to M*, if for each $(t_0, x_0) \in J \times \mathrm{int}_N M$ and $x(\cdot)$, solution of (1.1)–(1.2), we have $x(t) \in \mathrm{int}_N M$ for each $t \in [t_0, t_x[$ and there is $\delta = \delta(x) > 0$, such that

$$\liminf_{t \to t_x^-} \mathrm{dist}(x(t), \mathrm{fr}_N M) \leqslant \delta.$$

If $\delta > 0$ may be chosen independently of the solution $x(\cdot)$, we say that system (1.1) is *uniformly persistent in N with respect to M*. In the case of M compact, this latter definition is equivalent to that of *permanent coexistence* [28, 26] or *cooperativeness* [40, 22, 23], which requires that there is a compact subset of $\mathrm{int}_N M$ which attracts all the trajectories of system (1.1) with initial value in $\mathrm{int}_N M$.

Uniform persistence with respect to the positive cone $\mathbb{R}^d_+$ (first orthant) of $\mathbb{R}^d$ or with respect to the simplex $\{x \in \mathbb{R}^d_+ : \sum x_i = 1\}$ have been considered by several authors (see, for instance [15, 14, 40, 22, 25]) for their significance in the biological applications. In many examples coming from population dynamics, persistence for system (1.1) guarantees that none of the species involved comes into extinction, even if small stochastic perturbations are taken into account (see [5]). Uniform persistence is an even more robust concept and perhaps, so far the most suitable for modelling realistically situations of nonextinction of the species (see [5, 28, 24]).

As described above, these persistence-type concepts involve, essentially, two kind of conditions: flow-invariance of $\mathrm{int}_N M$ and repulsivity of $\mathrm{fr}_N M$ with respect to the solutions of (1.1) with values in $\mathrm{int}_N M$. For the former problem (flow-invariance) we may apply directly the results in [32] or those developed in [12] ensuring that no solution of (1.1)–(1.2) with $x_0 \in \mathrm{int}_N M$ touches the boundary $\mathrm{fr}_N M$ anywhere, so that we can focus our attention to the latter problem. To this end, it is useful to consider the situation (see also [25, 27])

$$\emptyset \neq G \subset N \text{ is a set } \textit{open} \text{ relatively to } N \tag{2.1}$$

$$S \subset N, \qquad G \cap S = \emptyset \tag{2.2}$$

and find conditions for the repulsivity of S with respect to the solutions of (1.1) lying in G. We also define

$$S^* := S \cap \mathrm{fr}_N G. \tag{2.3}$$

DEFINITION 2.1. Let $Z \subset N \backslash G$. We say that Z is *repelling with respect to* G if for each solution $x(\cdot)$ of (1.1) with $x(t) \in G$ for all $t \in [t_0, t_x[$, there are an open neighbourhood $A = A(x)$ of Z and $t_1 = t_1(x) \in [t_0, t_x[$ such that $x(t) \notin A$, for all $t \in [t_1, t_x[$. If, furthermore, the neighbourhood A of Z can be chosen independently of the considered solution $x(\cdot)$, we say that Z is *uniformly repelling with respect to* G. In the particular case that $Z = \{u\}$, with $u \notin G$, the point u is said to be *repulsive* (respectively, *uniformly repulsive*) *with respect to* G.

From the definition it is obvious to observe that any subset of a (uniformly) repelling set is (uniformly) repelling. A link with persistence-type conditions is shown in the following results.

PROPOSITION 2.1. *If S^* is compact and each point $u \in S^*$ is (uniformly) repulsive w.r. to G, then S is (uniformly) repelling. If S is compact and (uniformly) repelling w.r. to G, then (there is $\delta > 0$ such that)*

$$\liminf_{t \to t_x^-} \mathrm{dist}(x(t), S) > 0 \qquad (\geqslant \delta),$$

for each $x(\cdot)$ solution of (1.1) *with $x(t) \in G$, for all $t \in [t_0, t_x[$.*

PROPOSITION 2.2. *Let $M \subset N$ be with $\mathrm{int}_N M \neq \emptyset$ and $\mathrm{fr}_N M$ compact. Suppose that $\mathrm{int}_N M$ is flow-invariant with respect to Eq.* (1.1) *and each point of $\mathrm{fr}_N (\mathrm{int}_N M)$ is (uniformly) repulsive with respect to $\mathrm{int}_N M$. Then* (1.1) *is (uniformly) persistent in N with respect to M.*

The proof of Proposition 2.1 is just a straightforward application of the above definitions and the properties of compact sets, while Proposition 2.2 follows at once from the preceding one by the choice $G := \mathrm{int}_N M$ and $S := \mathrm{fr}_N M$. Hence these proofs are omitted.

A sufficient condition for the (uniform) repulsivity of a point $u \in \mathrm{fr}_N G$ is given in the following.

THEOREM 2.1. *Let $u \in \mathrm{fr}_N G$. Assume there are $\alpha \in J$, an open neighbourhood $\Omega' \subset \Omega$ of u and two continuous functions V and ψ, with $V := V(t, x) : [\alpha, b[\times \Omega' \to \mathbb{R}$ locally lipschitzian in x and $\psi : [\alpha, b[\times \mathbb{R}^- \to \mathbb{R}$, such that* (k$_0$) *the set* $K \subset \Omega'$, *defined by*

$$K := \{z \in \Omega' \cap \mathrm{fr}_N G : \limsup_{\substack{t \to b^- \\ G \ni x \to z}} V(t, x) = 0\}$$

is compact and $u \in K$,

(k$_1$) $$\lim_{\substack{t \to b^- \\ x \in G, \mathrm{dist}(x, K) \to 0}} V(t, x) = 0$$

(k$_2$) $\limsup_{\substack{t \to \tau^- \\ G \ni x \to z}} V(t, x) < 0$, *for all* $(\tau, z) \in [\alpha, b] \times (G \cap \Omega')$

(k$_3$) $\dot{V}(t, x) \leqslant \psi(t, V(t, x))$, *for all* $(t, x) \in [\alpha, b[\times (G \cap \Omega')$.

Then u is (uniformly) repulsive with respect to G provided that (there is $\delta > 0$ *such that)*

(l$_0$) ((l$_\delta$)) *for every* $k > 0$, *there is* $\eta = \eta_k > 0$ *such that for each* $\alpha \leqslant \tau < b$, *the problem*

$$w' = \psi(t, w), \qquad w(\tau) = -k \tag{2.4}$$

has a maximal solution $r = r(t)$, *with*

$$\sup_{t \geqslant \tau} r(t) \leqslant -\eta \qquad \textit{and} \qquad \liminf_{t \to t_r^-} r(t) < 0 \ (\leqslant -\delta).$$

Proof. We give a proof only in the case (more interesting for the applications) of the uniform repulsivity of the point u. The case of repulsivity may be treated by easy changes in the main argument and so it is omitted.

Fix an ε_1 such that $0 < \varepsilon_1 < \delta$ and let $\rho_0 > 0$ be such that $B[K, \rho_0] \subset \Omega'$ (as $K \subset \Omega'$ is compact). Using (k$_1$), we find $\beta_1 : \alpha \leqslant \beta_1 < b$ and $\rho_1 : 0 < \rho_1 \leqslant \rho_0$, such that

$$\inf\{V(t, x) : t \in [\beta_1, b[, x \in G \cap B[K, \rho_1]\} \geqslant -\varepsilon_1. \tag{2.5}$$

We claim now that there is $\beta_2 : \beta_1 \leqslant \beta_2 < b$ such that

$$-\varepsilon_2 := \sup\{V(t, x) : t \in [\beta_2, b[, x \in G \cap \mathrm{fr}\, B[K, \rho_1]\} < 0. \tag{2.6}$$

Indeed, by (k$_2$), we have $-\varepsilon_2 \leqslant 0$ (as $V(t, x) < 0$ on $[\alpha, b[\times (G \cap \Omega')$). By contradiction, suppose that $\varepsilon_2 = 0$. Then there is a sequence $(t_n, x_n) \in [\beta_1, b[\times (G \cap \mathrm{fr}\, B[K, \rho_1])$, with $t_n \to b$, such that $V(t_n, x_n) \uparrow 0$. Without loss of generality, we can assume $x_n \to z \in \mathrm{cl}_N\, G \cap \mathrm{fr}\, B[K, \rho_1] \subset \mathrm{cl}_N\, G \cap \Omega'$. By definition of K and since $z \notin K$, we get $z \in G$ and so, a contradiction with (k$_2$) is achieved. Hence the claim is proved.

Let $\eta = \eta_{\varepsilon_2} > 0$ be chosen according to condition (l_δ). Then, using again (k_1), we find $\beta_3 : \beta_2 \leqslant \beta_3 < b$ and $\rho : 0 < \rho < \rho_1$ such that

$$-\varepsilon_3 := \inf\{V(t, x) : t \in [\beta_3, b[, x \in G \cap B[K, \rho]\} > -\eta. \tag{2.7}$$

Finally, define $A := B(K, \rho)$, an open neighbourhood of u.

Let $x(\cdot)$ be a solution of (1.1)–(1.2) with $x(t) \in G$ for all $t \in [t_0, t_x[$. Since $\mathrm{cl}\, A = B[K, \rho] \subset \Omega$ is a compact set, we have that if $t_x < b$, then there is $t_1 \geqslant t_0$ such that $x(t) \notin A$, for all $t \in [t_1, t_x[$ (see [20, Theorem 2.1]). Hence we may assume, for the rest of the proof , that $t_x = b$.

Set $\gamma := \max\{t_0, \beta_3\}$.

If $x(t) \notin B[K, \rho_1]$ for all $t \in [\gamma, b[$, then for $t_1 = \gamma$, $x(t) \notin A$ for all $t \in [t_1, b[$ and we are done. So, suppose there is $\gamma_1 \geqslant \gamma$ such that $x(\gamma_1) \in B[K, \rho_1]$. We claim that there is $\gamma_2 > \gamma_1$ such that

$$x(\gamma_2) \notin B[K, \rho_1].$$

Indeed, suppose by contradiction that $x(t) \in B[K, \rho_1]$ for all $t \in [\gamma_1, b[$. Consider the function

$$v(t) := V(t, x(t)). \tag{2.8}$$

We have, for all $t \in [\gamma_1, b[$,

$$v(t) < 0 \qquad \text{(by } (k_2)), \tag{2.9}$$

$$v(t) \geqslant -\varepsilon_1 \qquad \text{(by (2.5))}, \tag{2.10}$$

$$D_+ v(t) \leqslant \psi(t, v(t)) \qquad \text{(by } (k_3)). \tag{2.11}$$

Let $r_1(t)$ be the maximal solution of

$$w' = \psi(t, w), \qquad w(\gamma_1) = v(\gamma_1)$$

according to hypothesis (l_δ).

By a comparison theorem [30, Theorem 1.4.1], we infer from (2.11) that $v(t) \leqslant r_1(t)$, for all $t \in [\gamma_1, t_{r_1}[$. Hence, from (2.10) and the choice of ε_1, we obtain

$$\liminf_{t \to t_{r_1}^-} r_1(t) \geqslant \liminf_{t \to t_{r_1}^-} v(t) \geqslant -\varepsilon_1 > -\delta$$

and a contradiction is achieved with respect to condition (l_δ). Therefore the claim is proved. (It is worthy to observe—see the proof of Theorem 2.2 below—that $t_{r_1} = b$, as $v(\cdot)$ is bounded from below and defined on $[\gamma_1, b[$.)

Thus $x(\gamma_2) \notin B[K, \rho_1]$ for some $\gamma_2 > \gamma_1$. If $x(t) \notin B(K, \rho_1)$ for all $t \geqslant \gamma_2$, then for $t_1 = \gamma_2$, $x(t) \notin A$ for all $t \geqslant t_1$ and the theorem is proved. So, suppose there is $\bar{t} > \gamma_2$ such that $x(\bar{t}) \in B(K, \rho_1)$. Define

$$\gamma^* := \sup\{t \in [\gamma_2, \bar{t}] : x(t) \notin B[K, \rho_1]\}.$$

We have $\gamma_2 < \gamma^* < \bar{t}$, $x(\gamma^*) \in G \cap \mathrm{fr}\, B[K, \rho_1]$ and $x(t) \in B[K, \rho_1]$ for $t \in [\gamma^*, \bar{t}]$. Consider again the function $v(\cdot)$ defined in (2.8). We have, for all $t \in [\gamma^*, \bar{t}]$, (2.9), (2.10), (2.11), and, by (2.6), $v(\gamma^*) \leqslant -\varepsilon_2$.

Let $r_2(t)$ be the maximal solution of

$$w' = \psi(t, w), \qquad w(\gamma^*) = -\varepsilon_2$$

according to hypothesis (1_δ). Then

$$r_2(t) \leqslant -\eta, \qquad \text{for all} \quad t \in [\gamma^*, t_{r_2}[. \tag{2.12}$$

By the above recalled comparison theorem, we infer that $v(t) \leqslant r_2(t)$ for all $t \geqslant \gamma^*$ and t belonging to the common domain of the functions. Now we note that $\bar{t} < t_{r_2}$. Otherwise, if $t_{r_2} \leqslant \bar{t} < b$, then, by (2.12)

$$\lim_{t \to t_{r_2}^-} r_2(t) = -\infty \qquad \text{and so} \qquad \lim_{t \to t_{r_2}^-} v(t) = -\infty,$$

contradicting the continuity of $v(\cdot)$ in t_{r_2}. Hence we may precise the above assertion to $v(t) \leqslant r_2(t)$ for all $t \in [\gamma^*, \bar{t}]$, and get, by (2.12), $v(\bar{t}) \leqslant -\eta$.

Finally, according to (2.7), we obtain

$$x(\bar{t}) \notin A.$$

As $\bar{t}$ is arbitrary, we have proved that for $t_1 := \gamma_2$,

$$x(t) \notin A, \qquad \text{for all} \quad t \geqslant t_1,$$

with A independent of the solution $x(\cdot)$ which has been considered. The proof is therefore complete. ∎

Remark 2.1. We note that if the functions $V(\cdot, \cdot)$ and $\psi(\cdot, \cdot)$ are defined with $\alpha = a$, then the assumptions in Theorem 2.1 also guarantee that the point u is *not reachable through* G; that is, there is no solution $x(\cdot)$ of (1.1)–(1.2) with $x_0 \in G$ such that $x(t_1) = u$ for some $t_1 \in I_x$ and $x(t) \in G$ for all $t \in [t_0, t_1[$ (see [12]). It is easily seen that G is flow-invariant provided that all the points of $\mathrm{fr}_N\, G$ are not reachable through G.

Remark 2.2. We observe that, instead of D_+, any other Dini derivative may be adopted in the proof. We also note that if V is differentiable in $G \cap \Omega'$, then the theorem holds true the same with

$$\dot{V}(t, x) = (\partial V/\partial t)(t, x) + ((\partial V/\partial x)(t, x) \mid f(t, x))$$

(see [46]). If V is only continuous, the situation is more delicate, like for stability theory. In this case, some argument should be adapted from [45]; however, such investigation has not been carried out in this paper.

A particular case in which Theorem 2.1 can be applied is when the function V can be chosen independent on the t-variable, i.e., $V = V(x)$. In this case, our hypotheses may be stated in a simpler form. In particular, (k_0), (k_1), (k_2) hold provided that $V: \Omega' \to \mathbb{R}$ verifies the conditions

(k_0') $K := \{x \in \Omega' \cap \mathrm{fr}_N G : V(x) = 0\}$ is compact and $u \in K$, and

(k_1') $V(x) < 0$, for all $x \in G \cap \Omega'$.

Even simpler assumptions have to be required whenever $G \subset \Omega'$. Namely, in this case, we can get the following variant of Theorem 2.1.

THEOREM 2.2. *Let $u \in \mathrm{fr}_N G$. Assume there are $\alpha \in J$, an open neighbourhood Ω' of u, with $G \subset \Omega' \subset \Omega$ and two continuous functions $\psi: [\alpha, b[\times \mathbb{R}^- \to \mathbb{R}$ and $V: \Omega' \to \mathbb{R}$, V locally lipschitzian, such that*

$$V(u) = 0 \quad \text{and} \quad V(x) < 0 \quad \text{for all} \quad x \in G$$

$$\dot V(t, x) \leqslant \psi(t, V(x)), \quad \text{for all} \quad (t, x) \in [\alpha, b[\times G.$$

Then u is (uniformly) repulsive with respect to G provided that the equation $w' = \psi(t, w)$ is (uniformly) persistent with respect to $\mathbb{R}^-$.

Proof. The proof follows the main lines of that of Theorem 2.1, anyhow we give a sketch of it for completeness. Again we consider only the case of uniform repulsivity, being the other one is completely similar.

Since equation $w' = \psi(t, w)$ is uniformly persistent with respect to $\mathbb{R}^-$, there is $\delta > 0$ such that

$$\limsup_{t \to t_r^-} r(t) \leqslant -\delta, \tag{2.13}$$

for each $r(\cdot)$ maximal solution of Eq. (2.4).

Fix an ε such that $0 < \varepsilon < \delta$ and let $\rho > 0$ be such that $B[u, \rho] \subset [V > -\varepsilon] := \{x \in \Omega' : V(x) > -\varepsilon\}$ and $\inf\{V(x) : x \in B[u, \rho]\} \geqslant -\varepsilon$. Let $x(\cdot)$ be any solution of (1.1)–(1.2), with $x(t) \in G$ for all $t \in [t_0, t_x[$. Arguing as in the proof of Theorem 2.1, we can assume, without loss of generality, $t_0 \geqslant \alpha$ and $t_x = b$. We want to prove that u is uniformly repulsive, with respect to G, with $A = B(u, \rho)$ (independent of $x(\cdot)$). Assume, by contradiction that there is a sequence $t_n \uparrow b$ such that $x(t_n) \in B(u, \rho)$ for every n and consider the function $v(t) := V(x(t))$ defined on $[t_0, b[$. As usual, we get $v(t) \leqslant r(t)$, for all $t \in [t_0, t_r[$, where $r(\cdot)$ is the maximal solution of (2.4) with $r(t_0) = v(t_0) < 0$. Moreover, from the continuity of $v(\cdot)$ and (2.13), we get that $r(\cdot)$ does not explode before b and so, $t_r = b$. Then

$$\limsup_{n \to +\infty} r(t_n) \geqslant \limsup_{n \to +\infty} v(t_n) \geqslant -\varepsilon > -\delta$$

and a contradiction with (2.13) is achieved. ∎

Again we remark that if $\alpha = a$, then u is not reachable through G.

Now we give an example of a function ψ satisfying the assumptions required in the preceding theorems. Suppose that $\psi : [\alpha, b[\times \mathbb{R}^- \to \mathbb{R}$ admits the factorization

$$\psi(s, z) := -\rho(s)\, \varphi(|z|) \tag{2.14}$$

with $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ and $\rho : [\alpha, b[\to \mathbb{R}$ continuous functions. Then, we have the following

Proposition 2.3. *Let $m > 0$ be such that*

(i$_1$) $\int_{t_1}^{t_2} \rho(s)\, ds \geqslant -m$, *for every* $t_1, t_2 \in [\alpha, b[$ *with* $t_1 < t_2$

and assume

(i$_2$) $\int_0^{u_0} (d\xi/\varphi(\xi)) = +\infty$ $(u_0 > 0)$.

Then, for any $k > 0$ and $\tau \geqslant \alpha$, there is $\eta = \eta_k > 0$ such that, for every $r(\cdot)$ (maximal) *solution of*

$$w' = -\rho(t)\, \varphi(|w|), \qquad w(\tau) = -k \tag{2.15}$$

we have

$$r(t) \leqslant -\eta, \qquad \textit{for all} \quad t \in [\tau, t_r[$$

and (*as a consequence*)

$$\limsup_{t \to t_r^-} r(t) \leqslant -\eta < 0.$$

Suppose further

(i$_3$) $\int_\alpha^b \rho(s)\, ds = +\infty$.

Then

$$\lim_{t \to t_r^-} r(t) = -\infty.$$

Proof. Let $r(\cdot)$ be the maximal solution of (2.15) and let $\beta \leqslant t_r \leqslant b$ be such that $r(t)<0$ for all $t\in[\tau,\beta[$, with β maximal. Since Eq. (2.15) has separable variables, we can write, for each $t\in[\tau,\beta[$,

$$\int_{|r(t)|}^{k}\frac{d\xi}{\varphi(\xi)}=\int_{\tau}^{t}\frac{r'(s)}{\varphi(|r(s)|)}\,ds=-\int_{\tau}^{t}\rho(s)\,ds,$$

that is,

$$\int_{|r(t)|}^{u_0}\frac{d\xi}{\varphi(\xi)}\leqslant -\int_{\tau}^{t}\rho(s)\,ds+v_k, \tag{2.16}$$

where $v_k:=|\int_k^{u_0}(d\xi/\varphi(\xi))|$, with $u_0>0$ fixed.

We define the function $\Phi(x):=\int_x^{u_0}(d\xi/\varphi(\xi))$ and observe that $\Phi\colon\mathbb{R}^+\to\,]-L,+\infty[$ $(L:=\int_{u_0}^{+\infty}(d\xi/\varphi(\xi))$, possibly $L=+\infty)$ is decreasing and

$$\liminf_{x\to 0^+}\Phi(x)=+\infty \qquad (\text{by } (\mathrm{i}_2)).$$

Using (2.16) and (i_1), we get

$$\Phi(|r(t)|)\leqslant m+v_k$$

and hence

$$r(t)\leqslant -\eta_k:=-\Phi^{-1}(m+v_k), \qquad \text{for all} \quad t\in[\tau,\beta[. \tag{2.17}$$

Therefore, $\beta=t_r$ and the first part of the thesis is proved.

If $t_r<b$, then $\lim_{t\to t_r^-}r(t)=-\infty$. If $t_r=b$, using (i_3) and taking the limits for $t\to b^-$ in both sides of (2.16), we get

$$\lim_{t\to b^-}\int_{u_0}^{|r(t)|}\frac{d\xi}{\varphi(\xi)}=+\infty$$

and therefore by (2.17), this implies $L=+\infty$ and so

$$\lim_{t\to b^-}|r(t)|=+\infty. \qquad \blacksquare$$

Even in the case of the decomposition (2.14), a possible variant of Proposition 2.3 can be obtained by restricting the hypotheses on ρ and relaxing the one on φ. Precisely, with a similar proof, we have

Proposition 2.4. *Replace* (i_1) *and* (i_2) *with*

(i_4) $\rho(s)\geqslant 0$, *for all* $s\in[\alpha,b[$.

Then the same conclusions of Proposition 2.3 *hold.*

Other possibilities, related to the decomposition (2.4) will not be considered in our applications.

3. Applications

As assumed before, let $N \subset \Omega$ be closed in Ω and flow-invariant and consider a set $M \subset N$ with $\operatorname{int}_N M \neq \emptyset$. Even if not explicitly assumed, the only interesting case is that $\operatorname{fr}_N M \neq \emptyset$. Otherwise uniform persistence occurs trivially.

In this section we apply the preceding theorems in order to obtain persistence and uniform persistence of system (1.1) in N, with respect to the set M. For $V: \Omega \to \mathbb{R}$, we set $[V=0] := \{x \in \Omega : V(x) = 0\}$.

Theorem 3.1. *Assume* $\operatorname{fr}_N M$ *is a compact set and suppose that for each* $u \in \operatorname{fr}_N (\operatorname{int}_N M)$, *there is a locally lipschitzian function* $V_u : \Omega \to \mathbb{R}$ *such that*

(v_1) $V_u(u) = 0$

(v_2) $V_u(x) < 0$, *for all* $x \in \operatorname{int}_N M$,

and there are a continuous function $\psi_u : J \times \mathbb{R}^- \to \mathbb{R}$ *and an open neighbourhood* Ω_u *of* $[V_u = 0] \cap \operatorname{fr}_N(\operatorname{int}_N M)$, *such that*

(v_3) $\dot{V}_u(t, x) \leqslant \psi_u(t, V_u(x))$, *for all* $(t, x) \in J \times (\Omega_u \cap \operatorname{int}_N M)$.

Then system (1.1) *is persistent* (*respectively, uniformly persistent*) *in* N, *with respect to* M, *provided that for each* u, *the equation* $w' = \psi_u(t, w)$ *fulfills* (l_0) (*respectively* (l_δ) *for some* $\delta = \delta_u > 0$) *with* $\alpha = a$.

Proof. We define the open set (with respect to N) $G := \operatorname{int}_N M$ and the compact sets $S := \operatorname{fr}_N M$ and $S^* = \operatorname{fr}_N (\operatorname{int}_N M)$. As a first step we note that G is flow-invariant for Eq. (1.1). In fact, by Remark 2.1, no point $S^* = \operatorname{fr}_N G$ is reachable through G. Then we observe that for each $u \in S^*$, the set $K_u := [V_u = 0] \cap \operatorname{fr}_N G \subset \Omega_u$ is compact and hence Theorem 2.1 ensures that each point $u \in S^*$ is repulsive (respectively, uniformly repulsive) with respect to $\operatorname{int}_N M$. Thus Proposition 2.2 provides the thesis. ∎

In the case that we can take, for the validity of (v_3), $\Omega_u = \Omega$ for each u, then an obvious variant of Theorem 3.1 may be obtained using Theorem 2.2. Precisely, we have that system (1.1) is persistent (respectively, uniformly persistent) in N, with respect to M, provided that for each u, the equation $w' = \psi_u(t, w)$ is persistent (respectively, uniformly persistent) with respect to $\mathbb{R}^-$.

A straightforward consequence of our results is the following.

COROLLARY 3.1. *Suppose that* $\mathrm{fr}_N M$ *is compact and let* $P\colon \Omega \to \mathbb{R}$ *be a differentiable function on* M *which satisfies*

$$P(x) = 0 \text{ for } x \in \mathrm{fr}_N M, \qquad P(x) > 0 \text{ for } x \in \mathrm{int}_N M.$$

Assume further that there is a neighbourhood Ω' *of* $\mathrm{fr}_N M$ *such that for every* $x \in \Omega' \cap \mathrm{int}_N M$ *and* $t \in J$,

$$\dot{P}(t, x) = (\nabla P(x) \mid f(t, x)) \geqslant P(x)\, \rho(t)$$

holds, where $\rho\colon J \to \mathbb{R}$ *is continuous and such that*

$$\int_a^b \rho(t)\, dt = +\infty \quad \text{and} \quad \int_{t_1}^{t_2} \rho(s)\, ds \geqslant -m > -\infty, \text{ for every } a \leqslant t_1 < t_2 < b.$$

Then system (1.1) *is uniformly persistent in* N *with respect to* M.

For the proof it is sufficient to combine Theorem 3.1, Proposition 2.3, and Remark 2.2, setting $V_u(x) = V(x) := -P(x)$ and $\psi_u(s, z) = \psi(s, z) := -\rho(s)\,|z|$.

Corollary 3.1 may be viewed as an analogue of Hofbauer's theorem [22, Theorem 1] (see also [25]) for the nonautonomous systems. Actually, when applied to the autonomous equations considered in [40, 22, 25], our theorem does not extend such results and in fact, it is less general, but it has the advantage of allowing time-dependent systems too and we recall that, as shown in [9], the results in [18, 22, 25] (which require conditions only for f on $\mathrm{fr}_N M$) are no more valid, in general, for nonautonomous systems. For instance, Corollary 3.1 might be useful in dealing with (1.1) when $f\colon \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ is p-periodic ($p > 0$) in the time variable. Such a situation is usually encountered in many differential equations models where seasonally varying coefficients are taken into account (see, for instance, [38, 34, 1]). In this case, it is a natural assumption to suppose that $\rho\colon \mathbb{R}_+ \to \mathbb{R}$ is p-periodic too. Then the hypotheses on ρ in Corollary 3.1 are satisfied provided that

$$\langle \rho \rangle := (1/p) \int_0^p \rho(s)\, ds > 0.$$

We present now some examples showing the range of application of Theorem 3.1. For simplicity, we restrict ourselves to the case $N = \Omega$.

EXAMPLE 3.1. Let $M \subset \Omega$ be a *convex body* (compact convex set with nonempty interior). Then, for each $u \in \mathrm{fr}_\Omega M = \mathrm{fr}\, M = \mathrm{fr}(\mathrm{int}\, M)$, there is a vector $\eta_u \neq 0$, the *outward normal* to M at u, such that $M \subset \{x \in \mathbb{R}^d\colon (x - u \mid \eta_u) \leqslant 0\}$ (see [43]). Defining, for each $u \in \mathrm{fr}\, M$,

$V_u(x) := (x - u \mid \eta_u)$, we have that (v_1) and (v_2) are fulfilled as $\operatorname{int} M \subset \{x \in \mathbb{R}^d : (x - u \mid \eta_u) < 0\}$ and, moreover, $\dot{V}_u(t, x) = (f(t, x) \mid \eta_u)$.

In particular, an easy consequence of Theorem 3.1 and Proposition 2.4 is the following.

Suppose that for each $u \in \operatorname{fr} M$ there are an outward normal η_u and a continuous function $\rho_u : J \to \mathbb{R}_+$ such that

$$(f(t, x) \mid \eta_u) \leqslant -\rho_u(t) \tag{3.1}$$

holds for all $t \in J$ and $x \in \operatorname{int} M$ in a neighbourhood of $\{x \in M: (x - u \mid \eta_u) = 0\}$. Then, system (1.1) is persistent with respect to M. Moreover, uniform persistence occurs whenever

$$\int_a^b \rho_u(t)\, dt = +\infty.$$

EXAMPLE 3.2. Let $W: \Omega \to \mathbb{R}$ be a C^1-function and let $M = [W \geqslant 0] := \{x \in \Omega : W(x) \geqslant 0\}$. Moreover, suppose that $\nabla W(x) \neq 0$ for each $x \in \operatorname{fr}_\Omega M$. Assume $\operatorname{fr}_\Omega M$ compact and $\operatorname{int}_\Omega M \neq \emptyset$. We note that, as a consequence of the above hypotheses, $\Omega' \cap [W > 0] = \Omega' \cap \operatorname{int}_\Omega M$, for a suitable neighbourhood Ω' of $\operatorname{fr}_\Omega M$. By the partition of unity, there is a C^1-function $V^*: \Omega \to \mathbb{R}$ such that $M = [V^* \leqslant 0]$, $\operatorname{fr}_\Omega M = [V^* = 0]$, and $\nabla V^*(x) = -\nabla W(x)$ on a neighbourhood of $\operatorname{fr}_\Omega M$ (see [9]). Defining, for each $u \in \operatorname{fr}_\Omega M$, $V_u(x) = V^*(x)$ (independent of u), we have that (v_1) and (v_2) are fulfilled and, moreover,

$$\dot{V}^*(t, x) = -(f(t, x) \mid \nabla W(x))$$

on a neighbourhood of $\operatorname{fr}_\Omega M$. Then, Theorem 3.1 gives the following consequence.

Suppose that there is a continuous function $\rho : J \to \mathbb{R}_+$ such that

$$(f(t, x) \mid \nabla W(x)) \geqslant \rho(t) \tag{3.2}$$

holds for all $t \in J$ and $x \in [W > 0]$ in a neighbourhood of $\operatorname{fr}_\Omega M$. Then, system (1.1) is persistent with respect to M. Moreover , uniform persistence occurs whenever

$$\int_a^b \rho(t)\, dt = +\infty.$$

In the particular case of the autonomous system, i.e., for $f(t, x) = f(x)$

(independent of t), $f: \Omega \to \mathbb{R}^d$ continuous, we consider the problem of persistence for

$$x' = f(x) \tag{3.3}$$

on $J = \mathbb{R}_+ = [0, +\infty[$. Then Examples 1 and 2 give the following.

Let $M \subset \Omega$ be a convex body and assume that for each $u \in \operatorname{fr} M$ there is an outer normal η_u such that

$$(f(x) \mid \eta_u) \leqslant 0 \qquad \text{for} \quad x \in \Omega_u \cap \operatorname{int} M,$$

with Ω_u a neighbourhood of $\{x \in M: (x - u \mid \eta_u) = 0\}$. Then Eq. (3.3) is persistent with respect to M. If

$$(f(x) \mid \eta_u) < 0, \qquad \text{for all} \quad x \in \{x \in M: (x - u \mid \eta_u) = 0\},$$

then Eq. (3.3) is uniformly persistent with respect to M.

Respectively, let $M = [W \geqslant 0]$, with $W: \Omega \to \mathbb{R}$ of class C^1 and $\nabla W(x) \neq 0$ for $x \in \operatorname{fr}_\Omega M$. Suppose also that $\operatorname{fr}_\Omega M$ is compact and $\operatorname{int}_\Omega M \neq \varnothing$.

If there is a neighbourhood Ω' of $\operatorname{fr}_\Omega M$ such that

$$(f(x) \mid \nabla W(x)) \geqslant 0 \qquad \text{for} \quad x \in \Omega' \cap [W > 0],$$

then Eq. (3.3) is persistent with respect to M. If

$$(f(x) \mid \nabla W(x)) > 0 \qquad \text{for all} \quad x \in \operatorname{fr}_\Omega M,$$

then Eq. (3.3) is uniformly persistent with respect to M.

This last result improves (in the autonomous case), [18, Theorem 3], since, compared with [18], we need $\operatorname{fr}_\Omega M$ compact and obtain uniform persistence, whence in [18], M is assumed to be compact and only weak persistence is achieved (see also [9]). We remark that, as shown in [9], Gard's theorem is not true in general for nonautonomous systems or when $\operatorname{fr}_\Omega M$ is not compact. From this point of view, our assumptions are rather sharp. Finally we note that our result for autonomous equations in the case of Example 3.2, could be also proved (for the part concerning uniform persistence) using [13, Corollary 1] or other general theorems about uniform persistence of dynamical systems (like, e.g., [25]). However, we stress the fact that our approach permits us to deal with nonautonomous equations too.

We end this section with a geometrical interpretation of conditions (v_1)–(v_3) in the case $N = \Omega$ and $\operatorname{fr}_\Omega M$ compact (so that $\operatorname{int}_\Omega M = \operatorname{int} M$).

Take, for $x \in \Omega$,

$$V(x) := -\mathrm{dist}(x, \mathbb{R}^d \backslash M)$$

and let, for each $u \in \mathrm{fr}_\Omega M$, $V_u \equiv V$.

Then (v_1) and (v_2) are satisfied and, for each u, $[V_u = 0] \cap \mathrm{fr}_\Omega(\mathrm{int}\, M) = \mathrm{fr}_\Omega(\mathrm{int}\, M)$. It is clear that

$$\begin{aligned} \dot{V}(t, x) &= \liminf_{h \to 0^+} \frac{\mathrm{dist}(x, \mathbb{R}^d \backslash M) - \mathrm{dist}(x + hf(t, x), \mathbb{R}^d \backslash M)}{h} \\ &= -\limsup_{h \to 0^+} \frac{\mathrm{dist}(x + hf(t, x), \mathbb{R}^d \backslash M) - \mathrm{dist}(x, \mathbb{R}^d \backslash M)}{h}. \end{aligned} \tag{3.4}$$

For brevity we restrict ourselves to the simplest case in which persistence is achieved choosing $\psi_u \equiv 0$ (according to Theorem 3.1 and Proposition 2.4). Thus we are looking for conditions ensuring that $\dot{V}(t, x) \leqslant 0$ on the points x of int M in a neighbourhood of $\mathrm{fr}_\Omega(\mathrm{int}\, M)$.

We define, for $\varepsilon > 0$,

$$M_\varepsilon := \{ z \in M : \mathrm{dist}(z, \mathbb{R}^d \backslash M) \geqslant \varepsilon \} \subset \mathrm{int}\, M.$$

Then, for any $y \in \mathbb{R}^d$, we have

$$\varepsilon = \mathrm{dist}(M_\varepsilon, \mathbb{R}^d \backslash M) \leqslant \mathrm{dist}(x + hy, \mathbb{R}^d \backslash M) + \mathrm{dist}(x + hy, M_\varepsilon). \tag{3.5}$$

So, suppose that $x \in \mathrm{int}\, M$ is fixed (in a suitable neighbourhood of $\mathrm{fr}_\Omega(\mathrm{int}\, M)$) and take $\varepsilon := \varepsilon_x = \mathrm{dist}(x, \mathbb{R}^d \backslash M)$. By (3.5), we easily get for $h > 0$,

$$\frac{\mathrm{dist}(x, \mathbb{R}^d \backslash M) - \mathrm{dist}(x + hy, \mathbb{R}^d \backslash M)}{h} \leqslant \frac{\mathrm{dist}(x + hy, M_\varepsilon)}{h}$$

and then, passing to the "lim inf" as $h \to 0^+$, we get $\dot{V}(t, x) \leqslant 0$, provided that

$$\liminf_{h \to 0^+} \frac{\mathrm{dist}(x + hf(t, x), M_\varepsilon)}{h} = 0 \tag{3.6}$$

(indeed, take $y = f(t, x)$ and use (3.4)).

We point out that condition (3.6) can be expressed as

$$f(t, x) \in T(M_\varepsilon; x) := \{ y \in \mathbb{R}^d : \liminf_{h \to 0^+} \mathrm{dist}(x + hy, M_\varepsilon)/h = 0 \},$$

where $T(M_\varepsilon; x)$ is the *Bouligand Contingent Cone* to M_ε at x (see [3]). Therefore, according to Theorem 3.1, we obtain the following.

Suppose there is $\varepsilon_0 > 0$ such that, for each $0 < \varepsilon < \varepsilon_0$,

$$f(t, x) \in T(M_\varepsilon; x), \qquad \text{for all} \quad t \in J \qquad \text{and} \qquad M_\varepsilon. \tag{3.7}$$

Then system (1.1) is persistent with respect to M.

Remark 3.1. Since f is continuous, the assumption (3.7) may be equivalently written substituting $T(\cdot\,;\cdot)$ with other tangent cones like, for instance, Bony's, Dubovickii–Miljutin's, or Clarke's tangent cones (see [3; 37; 8, Corollary 1.12; 47]). Moreover, by the Nagumo theorem [35], (3.7) is equivalent to the request that M_ε is *weakly flow-invariant for* $\varepsilon > 0$ small enough (see also [44]). We recall that M_ε is a weakly flow-invariant set for Eq. (1.1) if for any $(t_0, x_0) \in J \times M_\varepsilon$ there is at least one solution $x(\cdot)$ of (1.1)–(1.2) such that $x(t) \in M_\varepsilon$ for all $t \in I_x$. Hence we have proved that a sufficient condition in order that (1.1) be persistent with respect to M, is that int M is the union of a suitable collection $\{M_\varepsilon : 0 < \varepsilon < \varepsilon_0\}$ of *weakly* flow-invariant sets. From this point of view, the above result "extends" [18, Theorem 1] where it is supposed that f is autonomous and the sets whose union is int M are flow-invariant (on the other hand, in [18], an arbitrary collection of sets is allowed).

4. Ejective Boundary Points and Periodic Solutions

In this section, using the repulsivity results previously obtained, we present some application to the existence of nontrivial periodic solutions for the system

$$x' = f(t, x). \tag{4.1}$$

For simplicity, we suppose $f: \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and regular enough as to guarantee the forward uniqueness for the solutions to the Cauchy problems associated to Eq. (4.1). We also assume f p-periodic ($p > 0$) in the first variable, i.e., $f(t + p, x) = f(t, x)$, for every $x \in \mathbb{R}^d$ and $t \geqslant 0$.

For any $z \in \mathbb{R}^d$, we denote by $x(\cdot\,; z)$ the solution of (4.1) verifying the initial condition $x(0) = z$. Then, with these assumptions, the translation operator (Poincaré–Andronov map)

$$T: z \mapsto x(p; z)$$

is continuous on its domain

$$\operatorname{dom}(T) = \{z \in \mathbb{R}^d : x(\cdot\,; z) \text{ is defined in } [0, p]\},$$

and its fixed points are initial data for p-periodic solutions of Eq. (4.1) (see [29, 7]).

In many significant examples of periodic ordinary differential systems modeling infectious diseases transmission [2], competition of species [1, 21], and other ecological models [19], it is assumed that $f(t, 0) \equiv 0$; that is, $z=0$ is a fixed point of the map T.

We are interested in proving the existence of a nontrivial fixed point for T, that is, a nontrivial p-periodic solution of (4.1). To this end, we'll apply the Browder theorem [4, 36] on the existence of nonejective fixed points proving, at the same time, that the origin is repulsive.

For the reader's convenience, we recall the following

DEFINITION 4.1. Let C be a topological space, $z_0 \in C$, and $h: C \setminus \{z_0\} \to C$ a continuous map. We say that z_0 is an *ejective* point of h [36] if there exists an open neighbourhood A of z_0 such that for every $z \in A \setminus \{z_0\}$, there is a positive integer $m = m(z)$ such that $h^m(z)$ is defined and $h^m(z) \notin A$.

Then we have

THEOREM 4.1. *Let M be a compact convex flow-invariant set for Eq.* (4.1) *with $0 \in \operatorname{fr} M$ an extreme point of M. Assume* 0 *is not reachable through $M \setminus \{0\}$ and it is uniformly repulsive with respect to $M \setminus \{0\}$ in M. Then Eq.* (4.1) *has a* (*nontrivial*) *p-periodic solution $x(t)$, with $x(t) \in M \setminus \{0\}$, for all $t \geqslant 0$.*

Proof. As M is compact and flow-invariant, we have $M \subset \operatorname{dom} T$ and $T(M) \subset M$ (see [20]). We need to prove the existence of a fixed point for T in $M \setminus \{0\}$. First we note that $M \setminus \{0\}$ is flow-invariant, since 0 is not reachable through $M \setminus \{0\}$. Moreover, as 0 is uniformly repulsive with respect to $M \setminus \{0\}$ in M, we can find an open neighbourhood A of 0 such that for each $z \in M \setminus \{0\}$, there is $t_1 = t_1(z) \geqslant 0$ such that $x(t; z) \notin A$, for all $t > t_1$. Hence $T^m z$ is defined for any positive integer m and $T^m z \notin A$ if $mp \geqslant t_1$. Therefore we have proved that 0 is an ejective point of $T|_M$. Then Browder's theorem [4, Theorem 2; 36, Theorem 1.1]) applies and the thesis is achieved. ∎

Observe that if $0 \neq z^* \in M$ is a fixed point of T, then $z^* \notin A$ and hence, by the assumption of uniform repulsivity of 0, $x(t; z^*) \notin A$ for all $t \in \mathbb{R}_+$ (as $x(\cdot, z^*)$ is p-periodic).

Remark 4.1. The conditions required for 0 can be achieved using the results in [12, Sect. 3] and Section 3, taking $\Omega := \mathbb{R}^d$, $N := M$, $G := M \setminus \{0\}$, and $S = S^* := \{0\}$. In particular, a sufficient condition in order that 0 be not reachable through $M \setminus \{0\}$ is obtained by assuming

$f(t, 0) \equiv 0$ and that, for any $t_0 > 0$, $x(t) \equiv 0$ is the unique solution of (4.1) with $x(t_0) = 0$.

As an application of Theorem 4.1, we give a result of existence of non-trivial and non-negative periodic solutions. To this purpose, we introduce the following notation.

For $x = (x_1, ..., x_d) \in \mathbb{R}^d$, we set $x \geqslant 0$ (>0) if $x_i \geqslant 0$ (>0) for each $i = 1, ..., d$. Then $\mathbb{R}^d_+ := \{x \in \mathbb{R}^d : x \geqslant 0\}$. Let $\{e_1, e_2, ..., e_d\}$ be the canonical orthonormal basis in $\mathbb{R}^d$ and define $\hat{x}_i := x - x_i e_i = (x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_d)$.

Finally, let

$$\mathbf{1} := (1, 1, ..., 1) = \sum_{1}^{d}{}_i\, e_i.$$

THEOREM 4.2. *Let $g: [0, p] \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function such that $g(0, x) = g(p, x)$ for all x. Assume there is $R > 0$ such that, for each $t \in [0, p]$*

(α_1) $(g(t, x) \mid x) \leqslant 0$ *for* $x \geqslant 0$ *and* $|x| = R$,

and

(α_2) $g_i(t, \hat{x}_i) \geqslant 0$ *for* $\hat{x}_i \geqslant 0$, $|\hat{x}_i| \leqslant R$ *and* $i = 1, ..., d$.

Suppose there are r: $0 < r < R$, a continuous function $\Gamma: \mathbb{R}^+ \to \mathbb{R}^+$, and a Lebesgue integrable function $\Theta: [0, p] \to \mathbb{R}$, such that, for a.e. $t \in [0, p]$,

(α_3) $(g(t, x) \mid x) \geqslant \Gamma(|x|)\, \Theta(t)$ *for* $x \geqslant 0$ *and* $|x| \leqslant r$.

Then system

$$x' = g(t, x) \tag{4.2}$$

has at least one solution $x(\cdot)$ such that $x(0) = x(p)$, $x(t) \geqslant 0$, and $0 \neq |x(t)| \leqslant R$ for all $t \in [0, p]$, provided that

(α_4) $\int_0^p \Theta(s)\, ds > 0$

and

(α_5) $\int_0^{u_0} (\xi / \Gamma(\xi))\, d\xi = +\infty$ $(u_0 > 0)$

hold.

Proof. Without loss of generality, we can assume $r < R/2\sqrt{d}$ in (α_3). It can also be easily seen that there is a *continuous* function $\Theta_1: [0, p] \to \mathbb{R}$, with $\Theta_1(0) = \Theta_1(p)$, verifying (α_3) and (α_4). Let $f: \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ be the extension of g by p-periodicity in the first variable and $\tilde{\Theta}: \mathbb{R}_+ \to \mathbb{R}$ the

p-periodic extension of Θ_1. It is obvious that f fulfills (α_1), (α_2), and (α_3) (with $\tilde{\Theta}$ in place of Θ), for all $t \in \mathbb{R}_+$.

Assume for a moment the forward uniqueness for the solutions to Cauchy problems associated to equation $x' = f(t, x)$. (The general situation follows from this one by a perturbation argument and it is described at the end of the proof.)

First we observe that, as a consequence of (α_1), (α_2), the compact convex set

$$M_R := \{x \in \mathbb{R}^d : x \geqslant 0, |x| \leqslant R\}$$

is flow-invariant (see [12, Corollary 3; 42, Theorem 2.1]).

Moreover, $0 \in \operatorname{fr} M_R$ is an extreme point.

We prove now that, by (α_3)–(α_5), 0 is not reachable through $M_R \backslash \{0\}$ and it is uniformly repulsive with respect to $M_R \backslash \{0\}$ in M. To this end, we define the Liapunov-like function $V(x) := -|x|$, which is C^1 in $\mathbb{R}^d \backslash \{0\}$ and verifies

$$\begin{aligned}
&V(x) = 0 \qquad \text{if and only if} \quad x = 0\\
&V(x) < 0, \qquad \text{for all} \quad x \in M_R \backslash \{0\}\\
&\dot{V}(t, x) = (f(t, x) \mid \nabla V(x)) = -(f(t, x) \mid x)/|x|\\
&\qquad \leqslant -|x|^{-1} \Gamma(|x|)\, \tilde{\Theta}(t), \qquad \text{for all} \quad t \in \mathbb{R}_+ \text{ and } x \in (M_R \backslash \{0\}) \cap B(0, r).
\end{aligned}$$

Thus we can apply Theorem 2.1, Remark 2.1 (see also [12, Theorem 1]), and Proposition 2.3 with the choice $\rho(s) := \tilde{\Theta}(s)$ and $\varphi(\xi) := \Gamma(\xi)/\xi$. In fact, (α_5) implies

$$\int_0^{u_0} 1/\varphi = +\infty$$

while the periodicity of $\tilde{\Theta}$, together with (α_4), gives

$$\int_{t_1}^{t_2} \tilde{\Theta}(s)\, ds \geqslant -m$$

(uniformly with respect to $0 \leqslant t_1 \leqslant t_2$) and

$$\int_0^{+\infty} \tilde{\Theta} = +\infty.$$

Then Theorem 4.1 can be applied, providing the existence of an open neighbourhood A of 0 and a p-periodic solution $x(\cdot)$ of Eq. (4.1) (that is, a solution of (4.2) with $x(0) = x(p)$) such that

$$x(t) \in M_R \backslash A, \qquad \text{for all} \quad t \geqslant 0. \tag{4.3}$$

The general situation in which forward uniqueness for Cauchy problems is not assumed, can be dealt with an approximation argument analogous to those described in [11, Sect. 4; 8, Theorem 5.5]. Precisely, let us fix the point $z^* := (R/2\sqrt{d})\,\mathbf{1}$ and observe that, for any $x \in \mathbb{R}^d_+$, $|x| = R$,

$$(z^* - x \mid x) = (z^* \mid x) - R^2 = (R/2\sqrt{2}) \sum_{1}^{d}{}_i\, x_i - R^2 \leqslant (R^2/2) - R^2$$
$$= -R^2/2.$$

Let, for each $n \in \mathbb{N}$, $f_n: \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ be continuous, p-periodic in the first variable, with $(\partial f_n/\partial x)(t, x)$ continuous, and such that, for all $t \in \mathbb{R}_+ \times M_R$,

$$f(t, x) \leqslant f_n(t, x) \leqslant f(t, x) + (R/2nd)\,\mathbf{1} \tag{4.4}$$

holds (use the Stone–Weierstrass theorem).

Then, for each $n \in \mathbb{N}$, the function

$$h_n(t, x) := f_n(t, x) + (z^* - x)/n \tag{4.5}$$

verifies (α_1)–(α_3) and the uniqueness for the Cauchy problems. Thus, the first part of the proof applies and, for each $n \in \mathbb{N}$, we have a p-periodic solution $x_n(\cdot)$ to equation $x' = h_n(t, x)$, with $x_n(t) \in M_R \backslash A$, for $t \in \mathbb{R}_+$. Here the fact that the open set A may be chosen independently of n is crucial. This happens because each function h_n verifies (α_3) with the *same* Γ and Θ, provided that $r > 0$ is small enough (but fixed and independent of n, e.g., $0 < r < R/2\sqrt{d}$), and, from the proof of Theorem 2.1, the set A is found depending on the properties of the functions V and ψ (but not f).

Finally, the Ascoli–Arzelà theorem and the uniform convergence of h_n to f on $\mathbb{R}_+ \times M_R$ provide the existence of a subsequence of $(x_n)_n$ converging to a (nontrivial) solution $x(\cdot)$ of $x' = f(t, x)$, with $x(t) \in M_R \backslash A$ for all $t \geqslant 0$. The proof is therefore complete. ∎

Remark 4.2. Theorem 4.2 contains [39, Theorem 4.1]. Furthermore, various assumptions in [39, Theorem 4.1] are improved. Precisely, in [39], the author requires (α_1) (with a strict inequality), the condition

(H_1) $g(t, x) \geqslant -\alpha x$ on M_R $(0 \leqslant \alpha \leqslant 2/3)$,

which obviously implies (α_2) and

(H_3) for each $k: 0 < k < 2$ there is a neighbourhood of 0 in $\mathbb{R}^d_+$, where $g(t, x) \geqslant k a(t) x$, with $a(\cdot)$ non-negative and Lebesgue integrable on $[0, p]$.

Finally, a hypothesis on $a(\cdot)$ is assumed which implies $\int_0^p a(s)\,ds > 0$ [39, (4.3)].

We note that (H_3) gives, for any fixed k (e.g., $k = 1$) and $x \geqslant 0$, $(g(t, x) \mid x) \geqslant ka(t)|x|^2$ and so (α_3) and (α_5) of our Theorem 4.2 are satisfied with the choice $\Gamma(\xi) = |\xi|^2$ and $\Theta(t) := a(t)$. Hence our result proves that in Santanilla's theorem the non-negativity and the other assumptions on $a(\cdot)$ are unnecessary and it is sufficient to require for the function $a(\cdot)$, that

$$\int_0^p a(s)\,ds > 0.$$

Moreover, no condition on α and k (in (H_1), (H_3)), except $k > 0$ is needed.

A variant of Theorem 4.2 (extending also [17, Theorem 3.1]) may be produced if (α_3) is substituted by

$$(g(t, x) \mid x) \geqslant 0 \qquad \text{for} \quad x \geqslant 0, \quad |x| \leqslant r.$$

In this case, using Proposition 2.4 and an obvious perturbation argument, assumptions (α_4) and (α_5) can be avoided. Hence we find, with a different proof, a result previously obtained in [11, Remark 2].

Applying Theorem 4.1, various results of existence of nontrivial periodic solutions can be obtained just modifying the choice of the compact convex flow-invariant set for which 0 is an extreme point. For instance, we can take, for a fixed d-uple $(R_1, ..., R_d) > 0$

$$M := \{x \in \mathbb{R}^d_+ : x_i \leqslant R_i,\ i = 1, ..., d\}.$$

In this case, a possible variant of Theorem 4.2 can be derived just changing (α_1) with

$$(\alpha_1') \quad g_i(t, \hat{x}_i + R_i e_i) \leqslant 0, \qquad \text{for} \quad 0 \leqslant x_j \leqslant R_j\ (j \neq i).$$

On the other hand, the hypothesis of repulsivity for the origin may be achieved by other choices of the Liapunov-like function V. For instance, we could take (in the proof of Theorem 4.2)

$$V(x) := -(v \mid x), \qquad \text{with} \quad v > 0, \quad |v| = 1,$$

so that

$$\dot{V}(t, x) = -(f(t, x) \mid v).$$

Then the nonreachability and uniform repulsivity of the origin (w.r. to

$\mathbb{R}^d_+ \setminus \{0\}$) are guaranteed by assuming (α_4) and the existence of a vector $\eta > 0$, $|\eta| = 1$, such that

$$(\alpha_3') \quad (g(t, x) \mid v) \geqslant (\eta \mid x)\, \Theta(t) \text{ for } x \geqslant 0,\ |x| \leqslant r$$

holds for a.e. $t \in [0, p]$.

Other results can be produced along the line of [29, 41], assuming g of class C^1 in a neighbourhood of the origin, $g(t, 0) \equiv 0$, and suitable on the linear equation $y' = (\partial g/\partial x)(t, 0) \cdot y$.

Using Theorem 4.2 and its possible variants, as described above, applications can be easily found to the existence of nontrivial periodic solutions for the differential system

$$x_i' = x_i h_i(t, x) \qquad (i = 1, ..., d)$$

which is considered in various ecological applications (see, for instance, [38, 34, 19]).

Finally, we point out that the assumption $g(0, x) = g(p, x)$ can be dropped out if the periodicity on the derivative of the solution is not required and that our result may be also extended to the Carathéodory systems [20]. We also remark that, following [29, 41, 33, 10], further conditions on g can be produced for guaranteeing the uniqueness and the asymptotic stability of the positive periodic solution.

We end this section with an application of Theorem 4.1 to the system

$$\begin{aligned} x' &= x(a - bx - cy) \\ y' &= y(d - ex - fy), \end{aligned} \tag{4.6}$$

where the functions $a, b, c, d, e, f\colon \mathbb{R} \to \mathbb{R}$ are continuous and p-periodic.

THEOREM 4.3. *Assume that c, e are non-negative, but non-identically zero, and $b, f, \langle a \rangle, \langle d \rangle$ are positive. If condition*

$$(\langle d \rangle / \langle e \rangle) > \max(a/b); \qquad (\langle a \rangle / \langle c \rangle) > \max(d/f) \tag{4.7}$$

holds, then system (4.6) *has a nontrivial, p-periodic, and non-negative solution.*

Sketch of the Proof. Let $L_1, L_2 \in \mathbb{R}$ verify

$$(\langle d \rangle / \langle e \rangle) > L_1 > \max(a/b), \qquad (\langle a \rangle / \langle c \rangle) > L_2 > \max(d/f)$$

and set $M := [0, L_1] \times [0, L_2]$. It is not difficult to see that M is a compact flow-invariant set for system (4.6).

Since we have uniqueness for Cauchy problems associated to (4.6), using Liapunov-like functions and applying Theorem 2.4 in [48], one can see (just in the same way as in the last part of the proof of Theorem 4 in [48]) that the set $\{0\} \times [0, L_2]$ (resp. $[0, L_1] \times \{0\}$) is uniformly repelling with respect to $M \backslash \{0\} \times [0, L_2]$ (resp. $M \backslash [0, L_1] \times \{0\}$) in M, for system (4.6). Therefore, $(0, 0)$ is uniformly repulsive with respect to $M \backslash \{(0, 0)\} = (M \backslash \{0\} \times [0, L_2]) \cup (M \backslash [0, L_1] \times \{0\})$ in M.

So, we can apply Theorem 4.1 to the set M and the system (4.6). ∎

We remark that periodic solutions on the x- or y-axis are not excluded. However, with the hypotheses of Theorem 4.3 one can also guarantee the existence of a positive p-periodic solution for system (4.6) (use dissipativeness as in [48, Example 4.2]).

For f identically zero, we cannot apply Theorem 4.1 as we did in Theorem 4.3. In fact, if $\langle d \rangle$ is positive, no rectangle $[0, L] \times [0, L^*]$ exists so that it will be a flow-invariant set for system (4.6); if $\langle d \rangle$ is negative, the origin will not be repulsive in such a rectangle.

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